SUM 2019 MATH/STAT 394 Final

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Instructions

You have 2 hours to take the exam (DRS students excepted). You may use two regular-sized sheets of notes, front and back. You may use a calculator. You do not need a normal probability table. **Do not use the continuity correction for the normal approximation to the binomial–use the 68-95-99.7 rule.**

Useful Facts

- 68-95-99.7 rule: for a standard normal random variable Z, $\mathrm{P}(|Z| < 1) \approx .68$, $\mathrm{P}(|Z| < 2) \approx .95$, and $\mathrm{P}(|Z| < 3) \approx .997$.
- Jensen's Inequality: if g is convex on the range of X, then $\mathrm{E}[g(X)] \geq g(E[X])$.
- The function $\mathrm{sgn}(x)$ is 1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$.
- An upper bound x is tighter than an upper bound y if $x < y$, and similarly for lower bounds.

1 Continuous Random Variables

Suppose that I draw $U \sim \mathrm{Unif}(0, 1)$.

(a) Find the probability density function of $X = -\log(U)$. (5 points)

(b) Find $\mathrm{E}[X]$, the mean of X. (2 points)

(c) Let $Y = -\lambda \log(1 - U)$. Find the pdf and mean of Y. (3 points, with 2 bonus points if you use your answers from (a) and (b) rather than the density of U directly)

(d) Find $M_X(t)$ (be careful with the domain of this mgf), and use it to find $\mathrm{Var}(X)$. (5 points)

(e) Use Jensen's Inequality to provide a lower bound for the 3rd moment of X. (4 points)

(f) Let $V = 2U - 1$, and let $X' = \mathrm{sgn}(V)\log(|V|)$. Find the pdf and mean of X'. (4 points)

2 Gamblin'

Blackjack is a casino gambling game where you bet once and then play a round of the game. Your Blackjack strategy (which you use every round) leads to the following distribution of outcomes for a single round when you bet \$B:

x (outcome)	-\$B	\$0	\$B	\$1.5B
P(x)	0.475	0.25	0.075	0.2

You lose the \$B you bet with probability 0.475, you keep your \$B (for a net gain of \$0) with probability 0.25, etc.

(a) Assume that you bet \$1 every round. Use the central limit theorem/normal approximation to approximate the probability that you lose money (in aggregate) over the course of 100 hands of blackjack. **Round your calculation of σ^2 to the nearest tenth.** (10 points)

(b) You are immortal and you *really* like Blackjack, so you decide to stay in the casino and bet \$1 on Blackjack hands forever. What does the Law of Large Numbers say about the probability that you neither win nor lose money in aggregate (average winnings are zero), as you play Blackjack longer and longer? (5 points)

(c) Jeff Bezos is also immortal, uses your Blackjack strategy, and has access to an infinite supply of money. He joins your eternal Blackjack table, and after an initial bet of \$1, he doubles his bet each hand. Explain mathematically why, after any $n > 0$ hands of blackjack, Bezos still has a good chance of having turned a profit and thus the Law of Large Numbers does not apply to him. Name a critical assumption of the LLN that Jeff's betting scheme fails. (5 points)

3 Basketballin' Inequalities

The Dallas Mavericks, an average NBA basketball team, have a 50% chance of winning a game of basketball.

(a) Find an upper bound for the the probability that they win at least 65 games out of the next 100 games using the Markov Inequality. (5 points)

(b) You think that the outcomes of the basketball games are dependent–winning one game means the Mavericks are more likely to win the next, etc. Is the number of wins out of n games a Binomial random variable (and why or why not)? If the games are not independent, is your result from (a) valid? (4 points)

(c) From now on, **suppose that the games are independent**. Find a tighter upper bound for the probability of winning at least 65 out of 100 games than the one in part (a) by using symmetry. For any a, is it possible to find a Markov Inequality bound for $P(X \geq a)$ that is tighter than the symmetry bound for $P(X \geq a)$? (5 points)

(d) Use Chebyshev's Inequality to find an upper bound for this probability that is tighter than either of the bounds in the previous parts. (4 points + an extra 2 points if you get the tightest possible Chebyshev bound)

(e) Finally, use the normal approximation to the binomial distribution to provide an approximation of this probability. (5 points)

4 Counting

(a) Suppose we have three boxes (labelled A, B, and C), and n balls, numbered 1 to n. We distribute each ball uniformly at random among the three boxes.

(i) What is the probability of the event that at least one ball is placed into box A? (4 points)

(ii) What is the probability that no box is empty? (5 points)

(b) Two friends, Hari and Ram, went to buy lottery tickets. Unfortunately, there are only 10 lottery tickets left, numbered $1, \ldots, 10$. Hari and Ram pick up two tickets uniformly at random, numbered a and b, respectively. Consider the number $c = a \times b$. What is the probability that c is divisible by 15? (5 points)

(c) The product of Hari and Ram's ticket numbers, c, turns out to be divisible by 15. Their friend Sree walks in and chooses a ticket uniformly at random as well. What is the probability that her ticket number is between Hari and Ram's? (6 points)